Regularization on Ill-posed Source Terms in FEM Computation Using Two Magnetic Vector Potentials

Akihisa Kameari
Science Solutions International Laboratory, Inc.
2-21-7, Naka-cho, Meguro-ku, Tokyo, 153-0065, Japan
kamearia@ssil.com

Abstract – **In the formulation with two magnetic vector potentials, the inaccuracy of the Biot-Savart integration for the source term causes the divergence of the conjugate gradient method. The ill-posed equations are regularized by subtracting a rotational field from the numerically integrated field. The convergence is drastically improved. Even when the current continuity is not satisfied strictly we can get reasonable results.**

Index Terms - **finite element methods, edge elements, two potential method, reduced magnetic vector potential, ICCG convergence .**

I. INTRODUCTION

In electromagnetic analyses using the Finite Element Method, matching the discretization of complex-shaped source coils with the rest of the mesh is a time-consuming and difficult task. One method to avoid this burden is using two potentials (total and reduced potentials) in different regions and connecting the regions based on field continuity[1]. In this case, the coils can be defined independently from the FEM mesh. When the total and reduced magnetic vector potentials are used with edge elements, the accuracy of the Biot-Savart integration becomes essential for ensuring the well-posedness of the equations. One often encounters the divergence of the conjugate gradient method (ICCG) due to the ill-posedness of the equations in this formulation. Obviously, one can improve the convergence by increasing the accuracy of the Biot-Savart integration, but the computation for the integration becomes time-consuming and impractical in the engineering sense.

In this paper, a regularization method for the ill-posedness of the equations in the two potential method is proposed.

II. TWO POTENTIAL METHOD

In the two potential method, the total magnetic vector potential $\mathbf{A}_t$ and the reduced magnetic vector potential $\mathbf{A}_r$ are used for: regions (Ω_t) which include conductive and/or magnetic materials, and for the air region (Ω_r) surrounding the total potential regions, respectively. For the magnetic vector potentials, the edge element functional space is used.

In the total potential region, usual A formulation is adopted and the following equation is solved.

$$\nabla\times\frac{1}{m}\nabla\times\mathbf{A}_t + s\dot{\mathbf{A}}_t = 0 \tag{1}$$

Current sources embedded in the discretized mesh can also be included in this region, but for simplicity they are not considered in the formulation presented here. We can adopt here the A-ϕ formulation, that is equivalent to the A formulation when using edge elements and which is known to improve the convergence of the ICCG method.

In the reduced potential region, the equation

$$\nabla\times\frac{1}{m_0}\nabla\times\mathbf{A}_r = 0 \ , \tag{2}$$

is solved. At the interface (Γ_{tr}) of the two regions, the following continuity conditions are applied.

$$\mathbf{A}_t\times\mathbf{n} = (\mathbf{A}_r + \mathbf{A}_s)\times\mathbf{n} \tag{3}$$

$$\mathbf{H}_t\times\mathbf{n} = (\mathbf{H}_r + \mathbf{H}_s)\times\mathbf{n}\cdot \tag{4}$$

Here, $\mathbf{n}$ is the normal unit vector on the interface. $\mathbf{H}_t$, $\mathbf{H}_r$ and $\mathbf{H}_s$ are the magnetic intensities of the total, reduced and source field. $\mathbf{H}_t = 1/m\nabla\times\mathbf{A}_t$, $\mathbf{H}_r = 1/m_0\nabla\times\mathbf{A}_r$ and $\mathbf{H}_s$ is given by Biot-Savart integration of the source currents located outside of the total potential regions.

The Galerkin's weak form is derived from these equations as:

$$\int_{\Omega_t}\left(\nabla\times\mathbf{N}\cdot\frac{1}{m}\nabla\times\mathbf{A}_t + \mathbf{N}\cdot s\dot{\mathbf{A}}\right)dV + \int_{\Omega_r}\left(\nabla\times\mathbf{N}\cdot\frac{1}{m_0}\nabla\times\mathbf{A}_r\right)dV = \int_{\Gamma_{tr}}\mathbf{N}\times\mathbf{H}_s\cdot\mathbf{n}dS. \tag{5}$$

Here, $\mathbf{N}$ is the weighting function represented by the edge shape functions and includes the gradient of the scalar functions. The condition (3) is enforced strongly. Equation (5) must be satisfied when $\mathbf{N}$ is replaced by the gradient ∇w of any scalar function in the allowable functional space. Of course, the L.H.S. of the equation becomes zero, and the R.H.S. becomes,

$$\int_{\Gamma_{tr}}\nabla w\times\mathbf{H}_s\cdot\mathbf{n}dS = \int_{\Gamma_{tr}} w\nabla\times\mathbf{H}_s\cdot\mathbf{n}dS + \oint w\mathbf{H}_s dC \tag{6}$$

In the R.H.S. of (6), the line integration appears when the interface is open and becomes zero according to boundary conditions and the surface integration becomes zero when the source current does not flow through the interface. Thus,

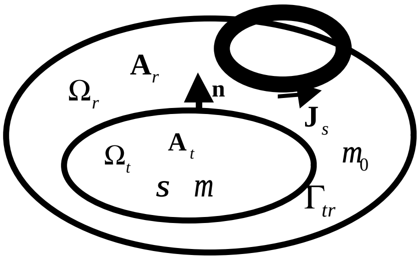

Fig. 1. Analysis regions for two potential method

ideally this substitution would make the R.H.S. of (5) to be zero, so that (5) is a non-definite equation as usual in the formulation using edge elements. But, when the integration of the R.H.S. of (5) is not strictly accurate, the equation becomes an ill-posed equation. In solving using the ICCG method, it converges to some extent according to the accuracy of the integration and finally diverges.

III. REGULARIZATION OF ILL-POSENESS

In general, the source field can be calculated by the Biot-Savart integration as,

$$\mathbf{H}_s = \int \frac{\mathbf{J}_s \times \mathbf{r}}{r^3} dV \cdot \qquad (7)$$

In case of simple coil geometry, analytical expressions can be used. However, for arbitrary coil shapes it is difficult and time-consuming to perform accurate integrations Rather, it would be preferable to be allowed to the accuracy of the integration.

The R.H.S. of (5) can be regularized as follows. The source field $\mathbf{H}_s$ on the interface is modified as,

$$\tilde{\mathbf{H}}_s = \mathbf{H}_s - \nabla \boldsymbol{j} \times \mathbf{n} \cdot \qquad (8)$$

The following equation is imposed:

$$\int_{\Gamma_{tr}} \nabla \boldsymbol{w} \times \tilde{\mathbf{H}}_s \cdot \mathbf{n} dS = \int_{\Gamma_{tr}} \nabla \boldsymbol{w} \times \mathbf{H}_s \cdot \mathbf{n} dS + \int_{\Gamma_{tr}} \nabla \boldsymbol{w} \times \mathbf{n} \cdot \nabla \boldsymbol{j} \times \mathbf{n} dS = 0 \qquad (9)$$

and solved for $\boldsymbol{j}$, and $\mathbf{H}_s$ in (5) is replaced with $\tilde{\mathbf{H}}_s$ .The regularization potential $\boldsymbol{j}$ is in the functional space of the nodal scalar functions on the interface. Equation (9) is a positive-definite symmetric equation and easily solved. This procedure corresponds to subtracting the rotational field component $\nabla \boldsymbol{j} \times \mathbf{n} = \nabla \times (\boldsymbol{j}\mathbf{n})$ from the calculated field.

IV. EXAMPLE

One example of the regularization is shown. We consider the model shown in Fig.2, where an iron core is magnetized by a coil. The coil is simulated by 4 rectangular parallelepipeds, overlapping on each other at sides. The current continuity is not satisfied strictly.

In Fig. 3, the convergence processes are shown for the cases with and without the regularization. Without the regularization, the ICCG method diverges after reaching an error of 3.5×10^{-3}. With the regularization, it converges to 1.3×10^{-14}, which probably is the numerical limit in the double precision. In Fig. 4, the distribution of the regularization potential $\boldsymbol{j}$ is shown. In this example, the magnetic fields resulted with and without the regularization are almost identical, and the difference is in the order of the minimum error in the ICCG iteration without the regularization.

V. Discussion

As shown in the example, using the regularization method we can get a convergent solution even when the source current continuity is not satisfied. We should note, though, that the solution obtained by regularization might still reflect the error caused by the inaccuracy of the source current condition. The author is expecting that the regularization is valuable in non-linear or transient analyses involving small perturbations. More test computations and validation of the method will be presented in the extended paper.

REFERENCES

[1] A. Kameari, "Solution of asymmetric conductor with a hole by FEM using edge-elements", *COMPEL, The International Journal for Computation and Mathematics in Electrical and Electronic Engineering*, Vol. 9, Supplement A, 1990, pp. 230-232.

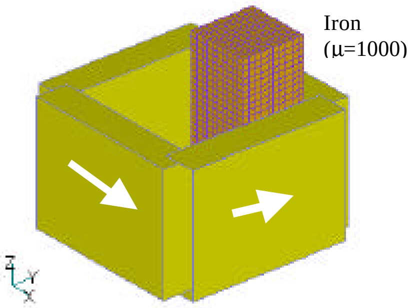

Fig. 2. Model for test calculation. The iron core is shown for 1/8 region.

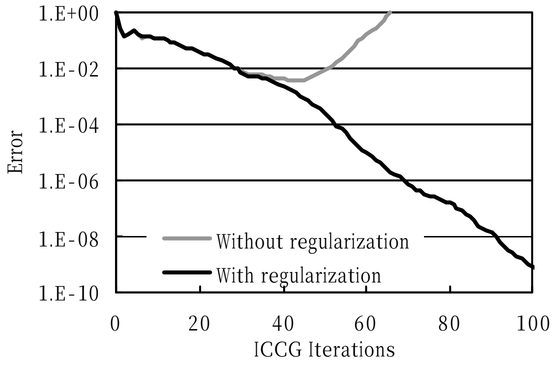

Fig. 3. Convergence of ICCG iterations with and without the regularization.

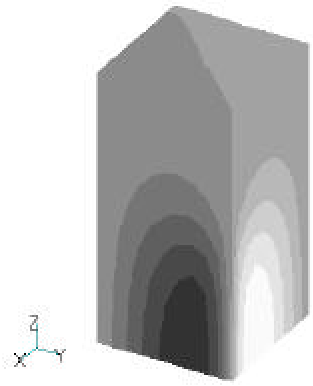

Fig. 4. Distribution of regularization potential